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Symmetry theory and Lagrangian inverse problem for time-dependent second-order differential equations

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Abstract. A set $\mathscr{X}_\Gamma$ of vector fields in the evolution space E playing the role of Newtonian vector fields, with respect to a second-order equation field Γ, is introduced and endowed with a $C^\infty(E)$-module structure. A dual set $\mathscr{M}^*_\Gamma$ is used for giving an answer to the Lagrangian inverse problem. The symmetry theory is also developed in this framework and, in particular, the characterisation of symmetries of Γ in terms of the transformation properties of the Lagrangian L is also given.

1. Introduction

A generalisation of the concept of infinitesimal symmetry of a regular Lagrangian L was proposed in a recent paper (Marmo and Mukunda 1986) in order to have a converse of the well known Noether's theorem for autonomous dynamical systems defined by a regular Lagrangian L. This generalisation allows us to make use of velocity-dependent transformations, and establishes in this way a one-to-one correspondence between first integrals of a second-order differential equation field Γ admitting a Lagrangian description and generalised infinitesimal symmetries of such a Lagrangian. A relevant role in the proof of this theorem is played by the set $\mathscr{X}_\Gamma$ (Sarlet *et al* 1984) associated with the second-order differential equation field Γ.

In this paper we will show how it is possible to enlarge this result to cover the case of a time-dependent Lagrangian dynamics. (Other alternative approaches to deal with time-dependent systems as being constrained systems can be found in Marmo *et al* (1983) and Cariñena and Ibort 1985a, b).) In order to do that we first generalise in §3 the set $\mathscr{X}_\Gamma$ to the case of Γ being a second-order differential equation vector field on the evolution space $E = \mathbb{R} \times TM$. We also generalise in §3 the $C^\infty(E)$-modules $\mathscr{M}^*_\Gamma$ and $\mathscr{X}^*_\Gamma$, two sets which have been shown to be very useful sets both in the treatment of dynamical symmetries of Γ, and in the so-called inverse problem of Lagrangian mechanics (Sarlet *et al* 1984, Cariñena *et al* 1989), and an application to the latter is given in §4. Finally in §5 we study the symmetries of Γ and give conditions for them to be Cartan symmetries. The possible relation with the existence of alternative Lagrangian descriptions and the determination of constants of the motion is also given.

2. Notation

Let M be an n-dimensional differentiable manifold. The evolution space $E = \mathbb{R} \times TM$ may be identified with the first jet bundle of (smooth) curves in M, $J^1(\mathbb{R}, M)$. Any vector field $\Gamma \in \mathscr{X}(E)$ such that $\langle \Gamma, \mathrm{d}t\rangle = 1$ and whose integral curves are all one-jet prolongation of curves on M is called a second-order differential equation field (SODE). They are distinguished by the following conditions:

$$\langle \Gamma, \mathrm{d}t\rangle = 1 \qquad \langle \Gamma, \theta^a\rangle = 0 \qquad a = 1, \ldots, n$$

where $\{\theta^a\}$ is a local basis of the set of contact 1-forms, $\theta^a = \mathrm{d}x^a - v^a\,\mathrm{d}t$, and t is the canonical coordinate of $\mathbb{R}$. In local coordinates (t, x^a, v^a) of the bundle $\mathbb{R} \times TM$, the local expression of such a vector field is

$$\Gamma = \frac{\partial}{\partial t} + v^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial v^a}$$

where $f^a \in C^\infty(E)$.

A remarkable fact is that the manifold E carries a type (1, 1) canonical tensor field S, called the vertical endomorphism (Crampin *et al* 1984, Saunders 1987), which is characterised by the following properties.

(i) S vanishes on vertical and SODE vector fields, and its images are vertical vectors.

(ii) $S(\partial/\partial t) = -\Delta$, where Δ denotes the Liouville dilation field of the vector bundle $E \to \mathbb{R} \times M$.

Its coordinate expression is

$$S = \frac{\partial}{\partial v^a} \otimes \theta^a.$$

The dual operator of S will be denoted S^*, i.e. $\langle X, S^*\alpha\rangle = \langle SX, \alpha\rangle$.

It has also been proved that if D is a SODE the following relations hold:

$$(\mathscr{L}_D S)(D) = 0$$
$$\mathscr{L}_D S \circ S = -S \circ \mathscr{L}_D S = S$$
$$(\mathscr{L}_D S)^2 = I - D \otimes \mathrm{d}t$$

where I is the identity tensor in E.

It is well known (see e.g. Crampin *et al* 1984, Sarlet and Cantrijn 1981) that, given a vector field

$$X = \sigma \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial x^a}$$

on $\mathbb{R} \times M$ there is an associated vector field $X^{(1)}$ on E, called its first prolongation, such that $X^{(1)}$ projects onto X and it preserves the distribution defined by the contact 1-forms θ^a. The coordinate expression of $X^{(1)}$ is

$$X^{(1)} = \sigma \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial x^a} + \xi^a \frac{\partial}{\partial v^a}$$

where $\xi^a = \dot{\eta}^a - v^a \dot{\sigma}$, and the notation $\dot{h}$, with h a function on the base $\mathbb{R} \times M$, means $\dot{h} = \partial h/\partial t + v^a \partial h/\partial x^a$.

3. The sets $\mathscr{X}_\Gamma$, $\mathscr{X}^*_\Gamma$ and $\mathscr{M}^*_\Gamma$

The fact that $X^{(1)}$ preserves the distribution defined by the contact 1-forms implies that, if D is a SODE, then $S(\mathscr{L}_{X^{(1)}} D) = 0$, or more specifically $\mathscr{L}_{X^{(1)}} D = -\dot{\sigma} D + V$, with

V being a vertical vector field, and then the (flow of the) first prolongation vector field $X^{(1)}$ transforms every SODE into another SODE, up to a time reparametrisation if $\dot{\sigma} \neq 0$.

The set $\mathscr{X}_\Gamma$ will be defined by means of this property but only for one specific SODE Γ, not necessarily for all SODE. That is, the elements of $\mathscr{X}_\Gamma$ are those vector fields $X \in \mathscr{X}(E)$ such that $S(\mathscr{L}_\Gamma X) = 0$.

Its local expression is given by

$$X = \sigma\Gamma + \eta^a \frac{\partial}{\partial x^a} + \Gamma(\eta^a)\frac{\partial}{\partial v^a} \qquad (\sigma,\ \eta^a \text{ local functions on } E)$$

from which it is evident that the vector field $X' = X + f\Gamma$ fulfils the same property for any function $f \in C^\infty(E)$. Actually

$$S\{\mathscr{L}_\Gamma(f\Gamma)\} = (\mathscr{L}_\Gamma f)S(\Gamma) + fS(\mathscr{L}_\Gamma\Gamma)$$

and the two terms on the right-hand side vanish. Thus it suffices to take only one element as a representative of the class $\{X + f\Gamma \mid f \in C^\infty(E)\}$, and in particular we choose the vector field X such that $\langle X, \mathrm{d}t\rangle = 0$. This choice implies that $[\Gamma, X]$ is not a SODE but a vertical vector field, because

$$\langle[\Gamma, X], \mathrm{d}t\rangle = \mathscr{L}_\Gamma\langle X, \mathrm{d}t\rangle + \langle X, \mathscr{L}_\Gamma \mathrm{d}t\rangle = \mathscr{L}_\Gamma\langle X, \mathrm{d}t\rangle = 0.$$

In the following we will denote by $\mathscr{X}_\Gamma$ the set

$$\mathscr{X}_\Gamma = \{X \in \mathscr{X}(E) \mid S(\mathscr{L}_\Gamma X) = 0 \quad \text{and} \quad \langle X, \mathrm{d}t\rangle = 0\}$$

where Γ is a given SODE field.

In much the same way as in the autonomous case, $\mathscr{X}_\Gamma$ can be endowed with a $C^\infty(E)$-module structure by means of the product

$$f \star X = fX + (\Gamma f)S(X) \qquad f \in C^\infty(E) \qquad X \in \mathscr{X}(E).$$

The map $\pi_\Gamma : \mathscr{X}(E) \to \mathscr{X}(E)$, given by $\pi_\Gamma(X) = X + S(\mathscr{L}_\Gamma X)$, is a morphism of $C^\infty(E)$-modules that is a projection map onto $\mathscr{X}_\Gamma$.

The set $\mathscr{M}_\Gamma^\star$ is defined in a similar way:

$$\mathscr{M}_\Gamma^\star = \{\mu \in \Lambda^1(E) \mid S^\star(\mathscr{L}_\Gamma \mu) = 0 \quad \text{and} \quad \langle\Gamma, \mu\rangle = 0\}.$$

The condition $S^\star(\mathscr{L}_\Gamma\mu) = 0$ requires μ to have the following local expression:

$$\mu = \rho\,\mathrm{d}t + \mu_a\theta^a + \Gamma(\mu_a)\psi^a \qquad (\text{with } \rho,\ \mu^a \text{ local functions on } E)$$

where $(\mathrm{d}t, \theta^a, \psi^a)$ is the local basis dual of that defined by the SODE field Γ, namely $(\Gamma, \partial/\partial x^a, \partial/\partial v^a)$. Moreover, the condition $\langle\Gamma, \mu\rangle = 0$ implies that $\rho = 0$. If a product by smooth functions $f \in C^\infty(E)$ is defined by

$$f \odot \mu = f\mu + (\Gamma f)S^\star(\mu) \qquad f \in C^\infty(E) \qquad X \in \mathscr{X}(E)$$

then $\mathscr{M}_\Gamma^\star$ becomes a $C^\infty(E)$-module and the map $\tau_\Gamma^\star : \Lambda^1(E) \to \Lambda^1(E)$, given by

$$\tau_\Gamma^\star(\alpha) = \alpha - S^\star(\mathscr{L}_\Gamma\alpha) - \langle\Gamma, \alpha\rangle \mathrm{d}t$$

is a morphism of $C^\infty(E)$-module onto $\mathcal{M}_\Gamma^\star$.

For the sake of completeness we will consider the set

$$\mathcal{X}_\Gamma^\star = \{ \phi \in \Lambda^1(E) \mid \mathcal{L}_\Gamma(S^\star\phi) = \phi \}$$

and in this case no additional restrictions are needed because if $\phi \in \mathcal{X}_\Gamma^\star$ then $\langle \Gamma, \phi \rangle = 0$. In fact, we have

$$\langle \Gamma, \phi \rangle = \langle \Gamma, \mathcal{L}_\Gamma(S^\star\phi) \rangle = \mathcal{L}_\Gamma \langle \Gamma, S^\star\phi \rangle - \langle \mathcal{L}_\Gamma \Gamma, S^\star\phi \rangle$$

and both terms vanish.

The product by smooth functions and the projection on $\mathcal{X}_\Gamma^\star$ are given respectively by

$$f \star \phi = f\phi + (\Gamma f) S^\star(\phi) \qquad \pi_\Gamma^\star(\phi) = \mathcal{L}_\Gamma(S^\star(\phi)).$$

The sets $\mathcal{X}_\Gamma^\star$ and $\mathcal{M}_\Gamma^\star$ are closely connected by $\mathcal{L}_\Gamma S^\star$:

$$\pi_\Gamma^\star = \mathcal{L}_\Gamma S^\star \circ \tau_\Gamma^\star \qquad \tau_\Gamma^\star = \mathcal{L}_\Gamma S^\star \circ \pi_\Gamma^\star$$

a relation which would not be possible without the aforementioned restrictions.

4. The Lagrangian inverse problem

The sets $\mathcal{X}_\Gamma^\star$ and $\mathcal{M}_\Gamma^\star$ may be used for establishing an alternative statement of the inverse problem theorem. We will say that a SODE Γ is Lagrangian if there exists $L \in C^\infty(E)$ such that $i_\Gamma \omega_L = 0$, where $\omega_L = -\mathrm{d}\Theta_L$ and Θ_L is the Poincaré–Cartan 1-form $\Theta_L = L\,\mathrm{d}t + S^\star(dL)$. The Lagrangian L does not need to be regular. Then the Lagrangian inverse problem theorem may be stated as follows.

Theorem. The three statements

(i) the SODE Γ is Lagrangian;

(ii) there exist a 1-form $\phi \in \mathcal{X}_\Gamma^\star$ and a function $f \in C^\infty(E)$ such that $\phi + f\,\mathrm{d}t$ is an exact 1-form;

(iii) there exists a closed 1-form $\alpha \in Z^1(E)$ such that $\mathcal{L}_\Gamma(\tau_\Gamma^\star(\alpha)) = 0$;

are equivalent.

Proof. First we prove that properties (i) and (ii) are equivalent. Let Γ be a Lagrangian SODE vector field, i.e. there exists $L \in C^\infty(E)$ such that $i_\Gamma \omega_L = 0$, or in an equivalent way $\mathcal{L}_\Gamma \Theta_L = \mathrm{d}L$ because $i_\Gamma \Theta_L = L$. If we take $\phi = \mathrm{d}L - (\mathcal{L}_\Gamma L)\,\mathrm{d}t$, since $\mathcal{L}_\Gamma\,\mathrm{d}t = 0$ and $S^\star(\mathrm{d}t) = 0$, we see that

$$\begin{aligned}
\mathcal{L}_\Gamma(S^\star(\phi)) - \phi &= \mathcal{L}_\Gamma(S^\star(\mathrm{d}L)) - \mathrm{d}L + \mathcal{L}_\Gamma(L\mathrm{d}t) \\
&= \mathcal{L}_\Gamma\{S^\star(\mathrm{d}L) + L\,\mathrm{d}t\} - \mathrm{d}L \\
&= \mathcal{L}_\Gamma \Theta_L - \mathrm{d}L = 0
\end{aligned}$$

and therefore $\phi \in \mathcal{X}_\Gamma^\star$ and the 1-form $\phi + (\mathcal{L}_\Gamma L)\,\mathrm{d}t$ is the exact 1-form $\mathrm{d}L$.

Conversely if $\phi \in \mathscr{X}^{\star}_{\Gamma}$ and there exists a function $f \in C^{\infty}(E)$ such that $\phi + f\,\mathrm{d}t = \mathrm{d}L$, then $i_\Gamma \mathrm{d}L = i_\Gamma\phi + f = f$ since $i_\Gamma\phi = 0$, and

$$0 = \mathscr{L}_\Gamma(S^\star(\phi)) - \phi = \mathscr{L}_\Gamma(S^\star(\mathrm{d}L)) - \mathrm{d}L + \mathscr{L}_\Gamma L\,\mathrm{d}t = \mathscr{L}_\Gamma\{S^\star(\mathrm{d}L) + L\,\mathrm{d}t\} - \mathrm{d}L$$

and therefore L is a Lagrangian (which might not be regular) for Γ.

Now we prove the equivalence of (i) and (iii). Let L be a Lagrangian for Γ and $\alpha \in B^1(E)$ a solution of the equation $i_\Gamma\alpha = L$. Then

$$\begin{aligned} 0 = \mathrm{d}L - \mathscr{L}_\Gamma\Theta_L &= \mathrm{d}i_\Gamma\alpha - \mathscr{L}_\Gamma(\langle\Gamma,\alpha\rangle\mathrm{d}t + S^\star(\mathscr{L}_\Gamma\alpha)) \\ &= \mathscr{L}_\Gamma\{\alpha - \langle\Gamma,\alpha\rangle\mathrm{d}t - S^\star(\mathscr{L}_\Gamma\alpha)\} = \mathscr{L}_\Gamma(\tau^\star_\Gamma(\alpha)) \end{aligned}$$

so that $\tau^\star_\Gamma(\alpha)$ is Γ-invariant.

Finally, let α be a closed 1-form such that $\mathscr{L}_\Gamma(\tau^\star_\Gamma(\alpha)) = 0$. If we take $L = i_\Gamma\alpha$ then

$$\tau^\star_\Gamma(\alpha) = \alpha - \langle\Gamma,\alpha\rangle\mathrm{d}t - S^\star(\mathscr{L}_\Gamma\alpha) = \alpha - \Theta_L$$

and

$$\mathscr{L}_\Gamma(\tau^\star_\Gamma(\alpha)) = \mathscr{L}_\Gamma\alpha - \mathscr{L}_\Gamma\Theta_L = \mathrm{d}L - \mathscr{L}_\Gamma\Theta_L.$$

then the function L is a Lagrangian for Γ.

Property (iii) provides us a geometrical interpretation of the results of Hojman *et al* (1983) avoiding the use of acceleration-dependent Lagrangians that, moreover, would be degenerate.

5. Symmetries and Noether's theorem

The sets $\mathscr{X}^\star_\Gamma$ and $\mathscr{M}^\star_\Gamma$ are also useful in the study of symmetries of the SODE Γ. Actually, the 2-form ω_L satisfies $i_\Gamma\omega_L = 0$, $i_V\mathscr{L}_\Gamma\omega_L = 0$ and $\omega_L(V, V') = 0 \quad \forall\, V, V' \in \mathscr{X}^V(E)$, so that the map $\hat{\omega}_L : \mathscr{X}(E) \to \Lambda^1(E)$ defined by contraction, i.e. $\hat{\omega}_L(X) = i_X\omega_L$ maps $\mathscr{X}_\Gamma$ onto $\mathscr{M}^\star_\Gamma$. Moreover, since $\mathscr{L}_\Gamma\omega_L = 0$ holds, $\hat{\omega}_L$ maps symmetries of Γ on Γ-invariant 1-forms in $\mathscr{M}^\star_\Gamma$, so that if L is regular there is a one-to-one correspondence between infinitesimal symmetries of Γ in the set $\mathscr{X}_\Gamma$ and Γ-invariant 1-forms in $\mathscr{M}^\star_\Gamma$.

Let X be a symmetry of Γ. If the 1-form μ_X is defined by $\mu_X = \hat{\omega}_L(X)$, then we get the following.

(i) If $\mu_X \in B^1(E)$, say $\mu_X = \mathrm{d}G$, then X is a Cartan symmetry (Prince 1983), i.e. $\mathscr{L}_X\omega_L = 0$, because $\mathscr{L}_X\Theta_L = \mathrm{d}(i_X\Theta_L - G)$.

(ii) If $\mu_X \notin B^1(E)$ but there exists a closed 1-form $\alpha \in Z^1(E)$ such that $\mu_X = \tau^\star_\Gamma(\alpha)$, then the function $L' = i_\Gamma\alpha$ is a Lagrangian subordinate (Marmo 1975) to L, $\omega_L = d\mu_X = \mathscr{L}_X\omega_L$. Under the assumption that L is regular, the relations

$$i_{R(X)}\omega_L = i_X\omega_{L'} \qquad \mathrm{d}t(R(X)) = 0 \qquad \forall X \in \mathscr{X}(E)$$

uniquely define a type (1, 1) tensor R, for which we symbolically write $R = (\hat{\omega}_L^{-1}) \circ (\hat{\omega}_{L'})$, i.e. it is Γ-invariant. Therefore $\mathrm{Tr}(R^k)$ are constants of the motion (Crampin 1983).

(iii) If neither (i) nor (ii) hold, we do not obtain any subordinate Lagrangian, but in the same way as in (ii), $\mathrm{Tr}(R^k)$ are constants of the motion where R stands now for $R = (\hat{\omega}_L^{-1}) \circ (\widehat{\mathrm{d}\mu_X})$.

Point (ii) explains the mechanism used in Sarlet (1983) to obtain dynamical symmetries. Let $\alpha \in Z^1(E)$ be a solution for $L = i_\Gamma \alpha$. Then we have seen that $\mu_L = \tau_\Gamma^\star(\alpha)$ is Γ-invariant and there is a uniquely defined dynamical symmetry X in $\mathscr{X}_\Gamma$ such that $\hat{\omega}_L(X) = \mu_L$. Thus, every Lagrangian for Γ is associated with a special dynamical symmetry of Γ. Moreover, for each pair of equivalent Lagrangians L_1, L_2 there exists an associated sequence of dynamical symmetries $\{R^k(X_1), R^k(X_2)\}$ where $R = (\hat{\omega}_{L_1}^{-1}) \circ (\hat{\omega}_{L_2})$ and $\hat{\omega}_{Li}(X_i) = \mu_{L_i}$, $i = 1, 2$. The four infinitesimal symmetries obtained by Sarlet (1983) are X_1, X_2, $R^{-1}(X_1)$ and $R(X_2)$.

Finally, we give a generalised Noether's theorem showing that a Cartan symmetry is, in this approach, the same thing as a Noether symmetry

Let G be a first integral of Γ. Since $\mathrm{d}G \in \mathscr{M}_\Gamma^\star$, there is one vector field $X \in \mathscr{X}_\Gamma$ such that $\hat{\omega}_L(X) = \mathrm{d}G$. Obviously $[\Gamma, X] = 0$. If the function F is defined by $F = G + i_X \Theta_L$ then $\mathscr{L}_X \Theta_L = \mathrm{d}F$ and when a contraction with the dynamical field is considered we will get

$$0 = i_\Gamma(\mathscr{L}_X \Theta_L - dF) = \mathscr{L}_X i_\Gamma \Theta_L - i_\Gamma dF = \mathscr{L}_X L - \mathscr{L}_\Gamma F.$$

Moreover, if we denote $X(D) = \pi_D(X)$ then, since the relation $X(D) = X + S[V, X]$ where $V = D - \Gamma$, we see that

$$\begin{aligned}
\mathscr{L}_{X(D)} L - \mathscr{L}_D F &= \mathscr{L}_X L - \mathscr{L}_\Gamma F + \mathscr{L}_{S[V,X]} L - \mathscr{L}_V F \\
&= i_{[V,X]}(\mathrm{d}L \circ S) - i_V \, \mathrm{d}F \\
&= i_V \mathscr{L}_X(\mathrm{d}L \circ S) - \mathscr{L}_X\{i_V(\mathrm{d}L \circ S)\} - i_V \, \mathrm{d}F \\
&= i_V\{\mathscr{L}_X(\mathrm{d}L \circ S) - \mathrm{d}F\}
\end{aligned}$$

and when taking into account that $i_V \, \mathrm{d}t = 0$ and $\mathscr{L}_X \, \mathrm{d}t = 0$ we will obtain

$$\mathscr{L}_{X(D)} L - \mathscr{L}_D F = i_V\{\mathscr{L}_X(\mathrm{d}L \circ S + L \, \mathrm{d}t) - \mathrm{d}F\} = i_V(\mathscr{L}_X \Theta_L - \mathrm{d}F) = 0.$$

Thus we can associate with each constant of motion G a unique Cartan symmetry $X(\Gamma)$ such that

$$\mathscr{L}_{X(D)} L - \mathscr{L}_D F = 0 \qquad \forall \, D \text{ SODE}.$$

Conversely, let $X \in \mathscr{X}(E)$ and $F \in C^\infty(E)$ be such that this relation holds. Without loss of generality we can assume that $X \in \mathscr{X}_\Gamma$. Then $G = i_X \Theta_L - F$ is a first integral of Γ because

$$\mathscr{L}_\Gamma(i_X \Theta_L - \mathrm{d}F) = \mathscr{L}_X L - \mathscr{L}_\Gamma F = 0.$$

Now subtracting $\mathscr{L}_{X(D)} L - \mathscr{L}_D F$ and $\mathscr{L}_X L - \mathscr{L}_\Gamma F$

$$\begin{aligned}
0 &= \mathscr{L}_{X(D)} L - \mathscr{L}_D F - (\mathscr{L}_X L - \mathscr{L}_\Gamma F) \\
&= \mathscr{L}_{S[V,X]} L - \mathscr{L}_V F \\
&= i_V\{\mathscr{L}_X(\mathrm{d}L \circ S) - \mathrm{d}F\} \\
&= i_V(\mathscr{L}_X \Theta_L - \mathrm{d}F)
\end{aligned}$$

so that $\mathscr{L}_X \Theta_L - \mathrm{d}F$ is a semibasic 1-form. But $\mathscr{L}_X \Theta_L - \mathrm{d}F = i_X \omega_L - \mathrm{d}G \in \mathscr{M}_\Gamma^\star$, and this is not possible except if it vanishes. Thus $\mathscr{L}_X \Theta_L - \mathrm{d}F = 0$ and X is a Cartan symmetry.

Notice that the condition $\mathscr{L}_{X(D)}L = \mathscr{L}_D F \quad \forall\, D$ SODE is actually two equations, $\mathscr{L}_{X(D_0)}L - \mathscr{L}_{D_0}F = 0$ with D_0 a particular SODE and $\mathscr{L}_{S[V,X]}L - \mathscr{L}_V F = 0$ for all vertical vector fields V. In coordinates

$$\eta^a \frac{\partial L}{\partial x^a} + \left(\frac{\partial \eta^a}{\partial t} + v^b \frac{\partial \eta^a}{\partial x^b}\right)\frac{\partial L}{\partial v^a} = \frac{\partial F}{\partial t} + v^b \frac{\partial F}{\partial x^b}$$

$$\left(\frac{\partial \eta^a}{\partial v^b}\right)\frac{\partial L}{\partial v^a} = \frac{\partial F}{\partial v^b}$$

where

$$X = \eta^a \frac{\partial}{\partial x^a} + \Gamma(\eta^a)\frac{\partial}{\partial v^a}.$$

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